# Intro to Motivic Homotopy Theory   $\mathbb{A}^1$ - htpy

Plan : ① Zariski & Nisnevich descent

② Motivic spaces.

③ Motivic Eilenberg - MacLane spaces.

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## §1. Descent Theory.

Assume the base scheme $S$ is qcqs & Noetherian.

$Sm_S$ := cat of smooth $S$-schemes of finite type.

guarantee $Sm_S$ is ess. small.

Recall A (Grothendieck) site is a cat equipped w/ a top. which is a choice of collection of families of maps $\{f_i : U_i \to X\}_{i \in I}$ a.k.a. coverings s.t. it satisfies

1. Base change   $g : X \to Y \rightsquigarrow \{U_i \times_X Y \to Y\}$.
2. Local character   $\{g_j : V_j \to X\}$ s.t. $\{V_j \times_X U_i \to U_i\}$ $\rightsquigarrow \{g_j\}$ covering
3. Identity   $\forall$ iso $\phi$. $\{\phi\}$ covering.

( Have to ask that $\mathcal{C}$ has pullbacks )

$\rightsquigarrow$ For $\mathcal{C} = Sm_S$. have different top :

- Zariski top   set-theoretic $X = \bigcup_i f_i(U_i)$

$\{U_i \xrightarrow{f_i} X\}$ open immersion. jointly surjective.

- étale top

  .. étale morphism . .. ..

- Nisnevich top

  .. étale , s.t. $\forall x \in X$ . $\exists$ $i$ and

  $y \in U_i$ s.t. $y \mapsto x$ induces iso on residue fields.

  $\mathcal{O}_{X,x}/m_x \xleftarrow[\cong]{f_i} \mathcal{O}_{U_i,y}/m_y$

**FACT** Zar $\leq$ Nis $\leq$ ét .

**Def** Consider the presheaf $F : \mathrm{Schs}^{op} \longrightarrow \mathcal{C}$ . Schs has a Grothendieck top $\tau$. Then $F$ is a $\tau$-sheaf if $\forall$ covering $\{f_i : U_i \longrightarrow X\} =: \mathcal{U}$

$$F(X) \xrightarrow{\sim} \lim_{\Delta} F(\mathcal{U}).$$

applying $F$ to Čech nerve $N(\mathcal{U})$.

**Rk** If $\mathcal{C}$ is 1-cat , then all higher info are forgotten. So replace $\Delta$ by $[0] \longrightarrow [1]$ . get ( equalizer )

$$F(X) \xrightarrow{\sim} \lim \left( \prod_i F(U_i) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{i,j} F(U_{ij}) \right)$$

If $\mathcal{C}$ is abelian , then recover the sheaf condition

$$0 \longrightarrow F(X) \longrightarrow \prod_i F(U_i) \xrightarrow{\alpha - \beta} \prod_{i,j} F(U_{ij})$$

left exact.

**Rk** Previous def is known as the descent condition. Actually for specific top , a small collection of covers is enough to decide the descent condition. instead of doing descent on all covers.

**Def** A cd-str ("completely decomposable") is a collection of comm. square in $\mathcal{C}$ closed under iso:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Given this, can define its top being the coarset top s.t. $\{B \to D, C \to D\}$ is a covering for every such square.

e.g. Zariski top is gen. by "distinguished Zar. square"

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd}$$

Nis. top is gen. by "distinguished Nis. square"

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

$i$ open immersion. $p$ restricts to iso $p^{-1}(X-U) \to X-U$.

**Def / Thm** 1) $F \in PSh^{\mathcal{C}}(Sch_S^{op})$ is a Zariski sheaf

$\Longleftrightarrow$ $F(\phi) = *$ and sends distinguished Zar. square to (htpy) pullback.

2) $F \in PSh^{\mathcal{C}}(Sm_S)$ is a Nis. sheaf $\Leftrightarrow$

$F(\phi) = *$ and sends distinguished Nis. sq. to (htpy) pullback.

## §2. Motivic Spaces

Working in $(Sm_S, Nis.)$. Let $F \in PSh^{\mathcal{C}}(Sm_S)$

<u>Def</u> 1. $F$ is $\mathbb{A}^1$-inv if $\forall X \in Sm_S$, $X \times \mathbb{A}^1 \longrightarrow X$ induces

$$F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$$

2. $F$ is strongly htpy inv if $\forall Y \to X$ Zariski loc. trivial affine morphism w/ fibers iso to affine space, then $F(X) \xrightarrow{\sim} F(Y)$

Write $PSh_{\mathbb{A}^1}(Sm_S)$ = full subcat of $PSh(Sm_S)$ gen. by 1.

$PSh_{ht}(Sm_S)$ = " " 2.

Then $PSh_{ht}(Sm_S) \subset PSh_{\mathbb{A}^1}(Sm_S) \subset PSh(Sm_S)$

strict !

**Def** The cat of motivic spaces is

$$\mathrm{Spc}(k) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k)$$

i.e. the full subcat of $\mathrm{PSh}(\mathrm{Sm}_k)$ s.t. objs are Nis. sheaves and are $\mathbb{A}^1$-inv.

**FACT** $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k)$

$$= \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{ht}(\mathrm{Sm}_k).$$

In general. hard to write down objs in $\mathrm{Spc}(k)$. Instead. there's a universal way to turn any presheaf on $\mathrm{Sm}_k$ into a motivic space:

$$\begin{tikzcd} \mathrm{PSh}(\mathrm{Sm}_k) \arrow[r, "L_{\mathrm{Nis}}"] \arrow[rr, bend right, "L_{mot}"'] & \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \arrow[r, "L_{\mathbb{A}^1}"] & \mathrm{Spc}(k) \end{tikzcd}$$

commute in $\infty$-cat sense

where $L_{\mathrm{Nis}}$ = sheafification w.r.t. Nis. top.

$L_{\mathbb{A}^1}$ = localization, a left adjoint to inclusion

$(L_{\mathbb{A}^1} : \mathrm{Shv}(\mathrm{Sm}_k) \longrightarrow \mathrm{Shv}_{\mathbb{A}^1}(\mathrm{Sm}_k))$

$L_{mot} \simeq L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}}$.

Explicitly. $L_{mot} = \mathrm{colim}(L_{\mathrm{Nis}} \to L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}} \to L_{\mathrm{Nis}} L_{\mathbb{A}^1} L_{\mathrm{Nis}} \to \cdots)$

computed in presheaf cat.

**Rk** $X \in Sm_k$ or $Sm_S$. $y(X)$ = assoc. presheaf of set by Yoneda. Then

$$L_{mot}(y(X)) = \text{motivic space of } X.$$

**Def** Pted motivic spaces $Spc(S)_*$ is given by

$$(-)_+ : Spc(S) \rightleftarrows Spc(S)_* : \text{Forget}$$

where $X_+ := X \sqcup S$

**Notation** ① cofiber of $Y \to X$ :

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/Y \end{tikzcd} \quad \text{pushout}$$

② $X \wedge Y := X \times Y / X \vee Y$

for $X \vee Y$ coproduct in $Spc(S)_*$

③ $\Sigma X :=$
$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd} \quad \text{pushout}$$

or $\Sigma X \simeq S^1 \wedge X$

**Rk** " $\simeq$ " in motivic spaces means $Y \xrightarrow{f} X$, then $L_{mot} f$ equiv. e.g. $\forall F \in PSh(Sm_S)$,

$$F \times \mathbb{A}^n \xrightarrow{\simeq} F$$

e.g. $\mathbb{P}^1_k = \mathbb{A}^1_k \cup \mathbb{A}^1_k$ intersection $= \mathbb{G}_m$.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook, "z \mapsto z"] \arrow[d, "z \mapsto 1/z"'] & \mathbb{A}^1_k \simeq * \arrow[d] \\ \mathbb{A}^1_k \arrow[r, hook] & \mathbb{P}^1_k \simeq \Sigma \mathbb{G}_m \end{tikzcd}$$

$$\mathbb{G}_m = \operatorname{Spec} k[z^{\pm 1}] = \mathbb{A}^1 \setminus \{0\}.$$

<u>Def</u> motivic spheres are

$$S^{1,1} := \mathbb{G}_m, \qquad S^{1,0} := S^1$$

$$S^{p,q} := S^{p-q} \wedge (\mathbb{G}_m)^{\wedge q}.$$

<u>FACT</u> ① $\mathbb{A}^n - \{0\} \simeq \Sigma^{n-1}((\mathbb{G}_m)^{\wedge n})$

pf.

$$\begin{tikzcd} \mathbb{G}_m \times \mathbb{G}_m \arrow[r, hook] \arrow[d] & \mathbb{G}_m \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \times \mathbb{G}_m \arrow[r, hook] & \mathbb{A}^2 - \{0\} \end{tikzcd}$$

collapse $\mathbb{A}^1$ and observe

$$\begin{tikzcd} \mathbb{G}_m \vee \mathbb{G}_m \arrow[r, hook] \arrow[d] & \mathbb{G}_m \arrow[d] \\ \mathbb{G}_m \arrow[r, hook] & * \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \mathbb{G}_m \wedge \mathbb{G}_m \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \mathbb{A}^2 - \{0\} \end{tikzcd}$$

$\Rightarrow \mathbb{A}^2 - \{0\} \simeq \Sigma(\mathbb{G}_m)^{\wedge 2}$. Keep going.

② $S^1 \simeq \mathbb{A}^1 / \{0, 1\}$.

pf.
$$\begin{tikzcd} S^0 \arrow[r, "(0,1)"] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$
pushout

③ as a corollary, $\mathbb{A}^n - \{0\} \simeq S^{2n-1, n}$,

$\mathbb{A}^n / \mathbb{A}^n - \{0\} \simeq S^{2n, n}$

$\mathbb{P}^n / \mathbb{P}^n - \{0\} \simeq S^{n, n}$

pf.
$$\begin{tikzcd} \mathbb{A}^n - \{0\} \arrow[r] \arrow[d] & \mathbb{P}^n - \{0\} \arrow[d] \\ \mathbb{A}^n \arrow[r] & \mathbb{P}^n \end{tikzcd}$$
pushout.

<u>Def</u> $\mathbb{A}^1$ - htpy gps

$X \in PSh(Sm_S)$. Then

$\pi_0 X :=$ Nis. sheaf assoc w/

$U \in Sm_S \longmapsto \pi_0(X(U))$

If $(X, x) \in PSh(Sm)_*$, then

$\pi_i(X, x) :=$ " "

$U \in Sm_S \longmapsto \pi_i(X(U), x)$

We write $\mathbb{A}^1$ - htpy gps to be:

$\pi_0^{\mathbb{A}^1} X := \pi_0(L_{mot} X)$

$\pi_i^{\mathbb{A}^1}(X, x) := \pi_i(L_{mot} X, x)$.

## § 3. Eilenberg - MacLane Spaces

One important example of motivic htpy thy is the Eilenberg MacLane space.

<u>Def</u> $\forall$ $A \in Ab_{Nis}(k)$ := cat of Nis. sheaves of abelian gps on $X \in Sm_k$.

denote

$$K(A, n) := DK(A[n]) \in PSh(Sm_k)$$

where $DK(A[n])$ = Dold - Kan construction of chain cpx w/ $A$ concentrated at deg $n$.

<u>Rk</u> Dold - Kan correspondence :

$$Ch_{\geq 0}(A_{Nis}(k)) \xrightarrow{\simeq} Fun(\Delta^{op}, Ab_{Nis}(k))$$

$$\text{chain cpx} \longmapsto \text{simplicial sheaf.}$$

Consider ( forget levelwise sheaf str )

$$\begin{array}{c} Fun(\Delta^{op}, Ab_{Nis}(k)) \subseteq Fun(\Delta^{op}, Fun(Sm_k^{op}, Ab)) \\ \downarrow \\ Fun(\Delta^{op}, Fun(Sm_k^{op}, Set)) \\ \downarrow \text{Forget} \\ PSh(Sm_k) \end{array}$$

Write $DK : Ch_{\geq 0}(Ab_{Nis}(k)) \longrightarrow PSh(Sm_k)$ be the composition.

**Prop** 1) $K(A,n) \in \mathrm{Shv}_{Nis}(Sm_k)$

2) $\pi_i K(A,n) = \begin{cases} A, & i=n \\ 0, & \text{else} \end{cases}$

3) $\exists$ nat. identification

$$\pi_0 \mathrm{Map}_{\mathrm{Shv}_{Nis}}(-, K(A,n)) \cong H^n_{Nis}(-, A).$$

**Rk** $K(A,n)$ not necess. a motivic space.

Examples ?

Note $K(A,n)(X) = \mathrm{Hom}_{\mathrm{PSh}_{Niv}(Sm_k)}(X, K(A,n))$

$= H^n_{Nis}(X, A)$.

if $A$ is $\mathbb{A}^1$-local.

e.g. ? $\mathbb{G}_a$.

X pretty worse. like BGL